Theoretical Mechanics
Final Exam
December 11, 2018

1. (20 Pts.) Consider three pendula, each of length l, that are coupled by identical springs with spring constant k. The pendula are horizontally separated by the natural rest length of the springs d. The outer pendula have a mass m and the center pendulum has a mass $2m$.
   a. Draw a suitable diagram for this problem, letting ϕ_i represent the angle of the pendula.

      The diagram should have three pendula hanging down. The angle the pendula are making with the vertical are the ϕ_i. The diagram describes the coordinate assignments for the rest of the problem.

   b. Assuming the angles ϕ_i remain small, show that the Lagrangian of this system is

$$L = \frac{1}{2}ml^2\dot{\phi}_1^2 + ml^2\dot{\phi}_2^2 + \frac{1}{2}ml^2\dot{\phi}_3^2$$
$$-\left(\frac{1}{2}mgl\phi_1^2 + mgl\phi_2^2 + \frac{1}{2}mgl\phi_3^2 + \frac{1}{2}kl^2\left(\phi_2 - \phi_1\right)^2 + \frac{1}{2}kl^2\left(\phi_3 - \phi_2\right)^2\right)$$

$$T = \frac{1}{2}mv_1^2 + \frac{1}{2}2mv_2^2 + \frac{1}{2}mv_3^2 = \frac{1}{2}ml^2\dot{\phi}_1^2 + ml^2\dot{\phi}_2^2 + \frac{1}{2}ml^2\dot{\phi}_3^2$$
$$U_{grav} = -mgl\cos\phi_1 - 2mgl\cos\phi_2 - mgl\cos\phi_3$$
$$\doteq -C + \frac{1}{2}mgl\phi_1^2 + mgl\phi_2^2 + \frac{1}{2}mgl\phi_3^2$$
$$U_{spring} = \frac{1}{2}k\left(\left(l\phi_2 - l\phi_1\right)^2 + \left(l\phi_3 - l\phi_2\right)^2\right)$$
$$L = T - U = \frac{1}{2}ml^2\dot{\phi}_1^2 + ml^2\dot{\phi}_2^2 + \frac{1}{2}ml^2\dot{\phi}_3^2$$
$$-\left(\frac{1}{2}mgl\phi_1^2 + mgl\phi_2^2 + \frac{1}{2}mgl\phi_3^2 + \frac{1}{2}kl^2\left(\phi_2 - \phi_1\right)^2 + \frac{1}{2}kl^2\left(\phi_3 - \phi_2\right)^2\right)$$

   c. Evaluate the mass and potential matrices and write the eigenvalue equation (do not attempt to solve).

$$M_{ij} = \frac{\partial^2 L}{\partial\dot{\phi}_i\partial\dot{\phi}_j} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$
$$K_{ij} = \frac{\partial^2 L}{\partial\phi_i\partial\phi_j} = \begin{bmatrix} mgl + kl^2 & -kl^2 & 0 \\ -kl^2 & 2mgl + 2kl^2 & -kl^2 \\ 0 & -kl^2 & mgl + kl^2 \end{bmatrix}$$
$$\det\left[M - \omega^2 K\right] = 0$$

2. (20 Pts.) Consider a canonical transformation generated by

$$S_2\left(q^1,\cdots,q^n,P_1,\cdots,P_n\right)=\sum_{i=1}^{n}q^iP_i+\varepsilon G\left(q^1,\cdots,q^n,P_1,\cdots,P_n\right)$$

where ε is an infinitesimal quantity.

a. By neglecting any order ε^2 or higher terms, show that the resulting canonical transformation differs from the identity transformation by terms of order ε with

$$P_i = p_i - \varepsilon\frac{\partial G}{\partial q^i}$$

$$Q^i = q^i + \varepsilon\frac{\partial G}{\partial P_i} = q^i + \varepsilon\frac{\partial G}{\partial p_i}$$

Specifically, why is the second equality in the Q^i equation valid?

The equations

$$P_i = p_i - \varepsilon\frac{\partial G}{\partial q^i}$$

$$Q^i = q^i + \varepsilon\frac{\partial G}{\partial P_i}$$

are just the rules for defining the new coordinates with an S_2 generating function and are exact. Nominally, they should be "solved" to obtain the transformation formulas $P_i\left(\vec{q},\vec{p}\right),Q_i\left(\vec{q},\vec{p}\right)$, leading to $G'\left(\vec{q},\vec{p}\right)=G\left(q^1,\cdots,q^n,P_1\left(\vec{q},\vec{p}\right),\cdots,P_n\left(\vec{q},\vec{p}\right)\right)$. By the chain rule

$$\frac{\partial G'}{\partial p_i}\left(q^1,\cdots,q^n,p_1,\cdots,p_n\right)=\sum_{j=1}^{n}\frac{\partial G}{\partial P_j}\left(\vec{q},\vec{P}\left(\vec{q},\vec{p}\right)\right)\frac{\partial P_j}{\partial p_i}\left(q^1,\cdots,q^n,p_1,\cdots,p_n\right)$$

$$=\frac{\partial G}{\partial P_i}\left(\vec{q},\vec{P}\left(\vec{q},\vec{p}\right)\right)-\varepsilon\sum_{j=1}^{n}\frac{\partial G}{\partial P_j}\left(\vec{q},\vec{P}\left(\vec{q},\vec{p}\right)\right)\frac{\partial}{\partial p_i}\left[\frac{\partial G}{\partial q^j}\left(\vec{q},\vec{P}\left(\vec{q},\vec{p}\right)\right)\right]$$

Clearly, retaining terms of only linear order in ε

$$Q^i = q^i + \varepsilon\frac{\partial G}{\partial P_i} = q^i + \varepsilon\frac{\partial G'}{\partial p_i}$$

Now, note that to the same order, by Taylor's theorem, $G'\left(\vec{q},\vec{p}\right)=G\left(q^1,\cdots,q^n,p_1,\cdots,p_n\right)$ with the new momenta replaced by the old in the original generating function.

b. Under this canonical transformation, show that the function $F\left(q^1,\cdots,q^n,p_1,\cdots,p_n\right)$ changes by an amount $dF\equiv F\left(Q^1,\cdots,Q^n,P_1,\cdots,P_n\right)-F\left(q^1,\cdots,q^n,p_1,\cdots,p_n\right)=\varepsilon\left[F,G\right]$ to linear order in ε where $\left[F,G\right]$ is the Poisson Bracket.

By part a. and Taylor's theorem, only collecting terms of linear order in ε ,

$$F\left(Q^1,\cdots,Q^n,P_1,\cdots,P_n\right) \doteq F\left(q^1,\cdots,q^n,p_1,\cdots,p_n\right)+\varepsilon\sum_{i=1}^{n}\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}-\varepsilon\sum_{i=1}^{n}\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}+\cdots$$

$$=\varepsilon\left[F,G\right]$$

c. If *G* is a constant of the motion of the Hamiltonian flow with Hamiltonian *H*, what is *dH*? What can you conclude? (Hint: Converse of Noether's Theorem)

A constant of the motion has vanishing Poisson bracket with the Hamiltonian. By part b., *dH*=0. The (infinitesimal) transformation generated by the constant of the motion leaves the Hamiltonian invariant. Incidentally, the same is true for finite transformations built up from a series of infinesimal ones.

3. (25 Pts.) Consider a string of uniform mass density σ with fixed end points and initial configuration

$$u\left(x=0,t\right)=0=u\left(x=L,t\right)$$

$$u\left(x,t=0\right)=f\left(x\right)=a\sin\left(\frac{3\pi}{L}x\right)$$

$$\frac{\partial u}{\partial t}\left(x,t=0\right)=0$$

a. Write down the Lagrangian of this system assuming a uniform tension τ in the string. Then use the Euler-Lagrange equation to derive the equation of motion for the string.

$$L=\frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2-\frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial\left(\partial L/\partial t\right)}\right]+\frac{\partial}{\partial x}\left[\frac{\partial L}{\partial\left(\partial L/\partial x\right)}\right]=\frac{\partial L}{\partial u}$$

$$\therefore\sigma\frac{\partial^2 u}{\partial t^2}-\tau\frac{\partial^2 u}{\partial x^2}=0$$

b. Introduce a linear damping force on the string. This change will modify the equation of motion to,

$$\sigma\frac{\partial^2 u}{\partial t^2}=\tau\frac{\partial^2 u}{\partial x^2}-\beta\frac{\partial u}{\partial t}$$

Explain why β must be a positive quantity.

If β were negative, the solution for $\partial u/\partial t$ would tend to grow exponentially with time indicating instability. Positive β insures that the string motion is damped.

c. Substitute a solution of the form $u\left(x,t\right)=\rho\left(x\right)\phi\left(t\right)$ and into the equation of motion in part b. Use separation of variables then the boundary and initial conditions to determine the eigenfunctions $\rho_n\left(x\right)$, and the space mode of the solution (don't solve for $\phi\left(t\right)$ yet).

$$u(x,t)=\rho(x)\phi(t)$$
$$\sigma\frac{\ddot{\phi}(t)}{\phi(t)}+\beta\frac{\ddot{\phi}(t)}{\phi(t)}=\tau\frac{\rho''(x)}{\rho}$$
$$\rho''(x)=-\alpha^2\rho \qquad \frac{\sigma}{\tau}\ddot{\phi}(t)+\frac{\beta}{\tau}\dot{\phi}(t)=-\alpha^2\phi(t)$$

The general solution for the $\rho(x)$ equation solving the boundary conditions in x is

$$\rho(x)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right) .$$

To solve the first initial condition only $n=3$ appears so

$$u(x,t)=a\sin\left(\frac{3\pi}{L}x\right)\phi(t) \qquad \ddot{\phi}(t)+\frac{\beta}{\sigma}\dot{\phi}(t)+\frac{\tau}{\sigma}\frac{9\pi^2}{L^2}\alpha^2\phi(t)=0$$

The initial conditions for the problem become $\phi(t=0)=1$ and $\dot{\phi}(t=0)=0$.

d. Show that $\phi(t)$ will have a functional form of $\phi(t)\propto e^{Ft}\left[\cos(Gt)-(F/G)\sin(Gt)\right]$, if

$$\beta^2<\frac{36\pi^2}{L^2}\sigma\tau.$$

You need not determine the coefficients *F*,*G*. [Hint: After separating variables in part c., assume $\phi(t)=e^{\gamma t}$ where γ is a constant.]

Using the exponential ansatz

$$\left[\gamma^2+\frac{\beta}{\sigma}\gamma+\frac{\tau}{\sigma}\frac{9\pi^2}{L^2}\right]=0\rightarrow\gamma=-\frac{\beta}{2\sigma}\pm\sqrt{\frac{\beta^2}{(2\sigma)^2}-\frac{\tau}{\sigma}\frac{9\pi^2}{L^2}}$$

If the condition is satisfied the square root is pure imaginary and the general solution is

$$\phi(t)=e^{Ft}\left[Ae^{iGt}+Be^{-iGt}\right]$$
$$\phi(0)=1\rightarrow A+B=1$$
$$\dot{\phi}(0)=0\rightarrow F(A+B)+iG(A-B)=0$$
$$A=\frac{1}{2}-\frac{F}{2iG}$$
$$B=\frac{1}{2}+\frac{F}{2iG}$$

A and *B* together give the solution form indicated.

4. (20 Pts.) We have discussed in class the solution of the wave equation for two point sources, located at $z=\pm d$, Problem 9.14. In the specific case that the sources are in phase, the far field radiated power (solid) angular distribution is

$$\frac{dP}{d\Omega}=\frac{P_0}{2\pi}\left[1+\cos\left((4\pi d/\lambda)\cos\theta\right)\right]$$

where P_0 is the power radiated by a single source, λ is the radiation wavelength, and θ is the usual polar angle with $\theta=0$ along the z axis.

a. Assume $\lambda = 4d$. This means there is one half wavelength change in the wave from one point source to the other. Calculate the locations θ that are maxima or minima in the power per unit solid angle.

$$\frac{d}{d\theta}\left[1+\cos\left(\pi\cos\theta\right)\right] = -\sin\left(\pi\cos\theta\right)\pi\left(-\sin\theta\right) = 0$$
$$\therefore \theta = 0, \pi/2, \pi$$

b. What are the values of the angular power at the maxima and minima? Explain physically.

$$\frac{dP_0}{d\Omega}\left(\theta = 0\right) = \frac{P_0}{2\pi}\left[1+\left(-1\right)\right] = 0$$
$$\frac{dP_0}{d\Omega}\left(\theta = \pi/2\right) = \frac{P_0}{2\pi}\left[1+\left(1\right)\right] = \frac{P_0}{\pi}$$
$$\frac{dP_0}{d\Omega}\left(\theta = \pi\right) = \frac{P_0}{2\pi}\left[1+\left(-1\right)\right] = 0$$

The sound field aligned or anti-aligned with the z axis vanishes because of destructive interference of the radiation from the two dipoles, whereas there is no relative phase shift for radiation in the $x-y$ plane. Here there is constructive interference.

Students also derived the conditions for constructive or destructive interference by setting $\cos\left(\pi\cos\theta\right) = \pm 1$. Although not as rigorous as above, this solution was acceptable because it yielded the correct answers.

c. Now assume $\lambda = d$. Calculate locations θ of angular power maxima and minima. How many maxima and minima are there? Explain. [Hint: $\cos\theta$ varies between 1 and -1 as θ varies between 0 and π .]

$$\frac{d}{d\theta}\left[1+\cos\left(4\pi\cos\theta\right)\right] = -\sin\left(4\pi\cos\theta\right)4\pi\left(-\sin\theta\right) = 0$$
$$\therefore \theta = 0, \cos\theta = 0, \cos\theta = \pm 1/4, \cos\theta = \pm 2/4 = \pm 1/2, \cos\theta = \pm 3/4, \cos\theta = \pm 1$$

These are the only solutions with $\cos\theta$ in the physical range. Therefore there are nine separate maxima and minima. By plugging these solutions into the power calculations one obtains

$$\frac{dP_0}{d\Omega}(\cos\theta = 0) = \frac{P_0}{2\pi}\left[1+(1)\right] = \frac{P_0}{\pi}$$
$$\frac{dP_0}{d\Omega}(\cos\theta = \pm 1/4) = \frac{P_0}{2\pi}\left[1+(-1)\right] = 0$$
$$\frac{dP_0}{d\Omega}(\cos\theta = \pm 1/2) = \frac{P_0}{2\pi}\left[1+(1)\right] = \frac{P_0}{\pi}$$
$$\frac{dP_0}{d\Omega}(\cos\theta = \pm 3/4) = \frac{P_0}{2\pi}\left[1+(-1)\right] = 0$$
$$\frac{dP_0}{d\Omega}(\cos\theta = \pm 1) = \frac{P_0}{2\pi}\left[1+(1)\right] = \frac{P_0}{\pi}$$

There is constructive interference in the even "$\cos\theta$" directions and destructive interference in the odd "$\cos\theta$". A ½ wavelength shift happens as each maxima becomes and minima and visa versa as θ is varied.

d. Suppose one has a single point source and a reflecting wall. How should one arrange the source to get the same wave field for $z > 0$ as in Problem 9.14?

By the method of images, simply put the single source at $\vec{r} = (0,0,d)$ and the wall along the $x - y$ plane. The sound field for $z > 0$ will be identical to the above.

5. (15 Pts.) In understanding both the wave equation and heat equation, the eigenfunctions of the three dimensional Helmholtz equation

$$\nabla^2\Phi + k^2\Phi = 0$$

are important.

a. Show the functions $\Phi_{\alpha,\beta,\gamma}(x,y,z) = e^{i\alpha x}e^{i\beta y}e^{i\gamma z}$ are eigenfunctions and compute the eigenvalue k in terms of α, β, and γ.

$$\nabla^2\Phi_{\alpha,\beta,\gamma} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\Phi_{\alpha,\beta,\gamma} = \left[-\alpha^2 - \beta^2 - \gamma^2\right]\Phi_{\alpha,\beta,\gamma}$$
$$\nabla^2\Phi_{\alpha,\beta,\gamma} + k^2\Phi_{\alpha,\beta,\gamma} = 0 \rightarrow k = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$$

b. What are the purely real eigenfunctions and associated eigenvalues whose values vanish at values $x = 0, a,\ \ y = 0, b,$ and $z = 0, c$? What is the frequency of the lowest non-zero mode of a cube having $b = c = a$?

The general x eigenfunction is $Ae^{iax} + Be^{-i\alpha x}$. To satisfy the boundary condition at $x = 0$, $A + B = 0$. To solve the boundary condition at $x = a$

$$Ae^{i\alpha a} + Be^{-i\alpha a} = 2Ai\sin\alpha a = 0 \rightarrow \alpha = n\pi / a.$$

This works for all positive integers n. $n = 0$ is excluded because then the solution vanishes. Clearly, the same argument works in the y and z directions. So the three-dimensional eigenfunctions satisfying the boundary conditions are

$$\sin(m\pi x / a)\sin(n\pi x / a)\sin(p\pi x / c) \qquad m, n, p = 1, 2, 3, \cdots$$

The eigenvalues are

$$k = \sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + \frac{p^2\pi^2}{c^2}}$$

For the cube the lowest eigenfrequency is $\omega_{111} = \sqrt{3}\pi / a$.

c. What are the purely real eigenfunctions and associated eigenvalues whose derivatives vanish at values $x = 0, a, \quad y = 0, b,$ and $y = 0, c$?

To make the derivative boundary conditions vanish simply change the sines to cosines. Here zero is a possible eigenvalue, and if all three m, n, p are zero one obtains the constant function, which indeed is an eigenfunction.